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# SOME PROPERTIES OF REPETENDS.

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1. In the present paper I investigate some properties of repetends arising from the development of the fraction $M/N$, where $N$ is not generally a prime number, but $M$ and $N$ are any whole numbers prime to each other. The investigation will not be confined to the decimal system of notation, and when a number is expressed in any notation other than the decimal, it will be enclosed in brackets, with the base expressed in the decimal notation as a subscript. For example, $15 = [17]_8 = [13]_{12}$. When all the numbers in any investigation have been expressed in the adopted notation, the subscripts may be omitted; otherwise when no subscript is expressed, the decimal notation will always be understood. The base will generally be denoted by the letter $B$.

2. The number of numbers less than and prime to a given number $k$, has been called by Prof. Sylvester the totient of $k$, and he proposes for it the notation $\tau(k)$. Each of the numbers whose enumeration makes up the totient may be called a totitive of $k$. If $k = a^t b^u \ldots h^z$ we know from the Theory of Numbers that

$$\tau(k) = a^{t-1} b^{u-1} \ldots h^{z-1} (a-1)(b-1) \ldots (h-1).$$

3. LEMMA.—If $A$ and $k$ are prime to each other, the difference $A^{\tau(k)} - 1$ is divisible by $k$; and if $m$ is the least number for which $A^m - 1$ is divisible by $k$, then $m$ is a divisor of $\tau(k)$.

Let $A$ be any one of the totitives of $k$, and form the series,

$$1, A, A^2, A^3, \ldots \qquad (1)$$

Since the terms of this series when divided by $k$ cannot give more than $\tau(k)$ different remainders, we must have, on continuing the series, a term congruous to some one of the preceding terms; suppose that $A^m$ is the first term which possesses this property. We must have

$$A^m \equiv 1, \quad \text{mod. } k.$$

For suppose $A^m \equiv A^n, \quad \text{mod. } k.$

$$A^m - A^n = A^n (A^{m-n} - 1) \equiv 0, \quad \text{mod. } k;$$

$$\therefore \quad A^{m-n} - 1 \equiv 0, \quad \text{mod. } k.$$

That is $A^m$ is not, as supposed, the first term congruous to a preceding term.

If the remainders of the series,

$$1, A, A^2, A^3, \ldots A^m, \tag{2}$$

include all the totitives of $k$, we have $m = \tau(k)$.

$$\therefore \quad A^{\tau(k)} - 1 \equiv 0, \quad \text{mod. } k.$$

If $\tau(k) > m$, let $B$ be a totitive of $k$ not included in the series (2), and form the series,

$$B, BA, BA^2, BA^3, \ldots BA^m. \tag{3}$$

No one of the terms of series (3) can be congruous to any one of series (2); for suppose

$$BA^n \equiv A^p, \quad \text{mod. } k;$$

$$\therefore \quad B \equiv A^{p-n}, \quad \text{mod. } k,$$

which is contrary to the hypothesis. Neither can two terms of series (3) be congruous; for suppose

$$BA^n \equiv BA^p, \quad \text{mod. } k;$$

$$\therefore \quad A^n \equiv A^p, \quad \text{mod. } k,$$

which is contrary to the hypothesis, $p$ and $n$ being both less than $m$. Therefore series (3) includes $m$ totitives of $k$; therefore $\tau(k) = 2m$ or $\tau(k) > 2m$.

If $\tau(k) > 2m$, let $C$ be a totitive of $k$ not included in series (2) or (3), and form the series,

$$C, CA, CA^2, CA^3, \ldots CA^m. \tag{4}$$

By reasoning upon series (4) as upon series (3), we can show that we have $m$ totitives of $k$, and therefore $\tau(k) = 3m$, or $\tau(k) > 3m$.

Finally, when all the totitives of $k$ are exhausted,

$$\tau(k) = qm.$$

And since $A^m \equiv 1, \quad \text{mod. } k,$

$$A^{qm} = A^{\tau(k)} \equiv 1, \quad \text{mod. } k.$$

4. THEOREM I.—If the base of notation contain all the prime factors of $N$, the fraction $M/N$ reduces to an exact decimal, and the number of places is equal

to the greatest integer in $(n + n' - 1) / n'$, where $n - n'$ is the greatest difference found by subtracting the exponent of each prime factor in $N$ from the exponent of the same factor in the base.

Let $N = a^t b^u \ldots h^z$, $B = \lambda a^{t'} b^{u'} \ldots h^{z'}$, $i =$ the greatest integer in $(n + n' - 1) / n'$. Then, since $in' = n$, or $in' > n$, and $(i - 1) n' < n$, we have $B^i$, but not $B^{i-1}$, divisible by $N$;

i. e.
$$\frac{MB^i}{N} = q.$$

5. THEOREM II.—If the base contain no prime factor of $N$ the fraction $M / N$ reduces to a repetend.

For by the lemma,
$$B^{\tau(N)} - 1 \equiv 0, \quad \text{mod. } N;$$
$$\therefore \quad \frac{MB^{\tau(N)}}{N} = q + \frac{M}{N}.$$
That is, after $\tau(N)$ places the figures repeat.

6. THEOREM III.—If the base contain some but not all the prime factors of $N$, the fraction $M / N$ reduces to a mixed decimal, and the number of places in the non-repeating part is determined as in Theorem I.

Let
$$N = a^t b^u \ldots f^w h^z, \; B = \lambda a^{t'} \ldots h^{z'}.$$
We have
$$\frac{M}{N} = \frac{MB^{m+i}}{a^t b^u \ldots f^w h^z} = \frac{MM'B^m}{b^u \ldots f^w},$$
where $M' = B^i \div a^t \ldots h^z$, and $i$ has the same meaning as in Theorem I.

In what follows we suppose the base prime to $N$.

7. THEOREM IV.—The number of places in the period of the repetend $M / N$ is a divisor of $\tau(N)$.

For if
$$\frac{MB^n}{N} = q + \frac{M}{N},$$
we have
$$M(B^n - 1) \equiv 0, \quad \text{mod. } N,$$
or since $M$ is prime to $N$,
$$B^n - 1 \equiv 0, \quad \text{mod. } N.$$
Therefore by the lemma, $n$ is a divisor of $\tau(N)$.

EXAMPLES:—
$$\tfrac{1}{63} = .\dot{0}1587\dot{3},$$
$$\tau(63) = 3(3 - 1)(7 - 1) = 36.$$
The number of places is 6, a divisor of $\tau(63)$.

$$\tfrac{1}{77} = [\tfrac{1}{65}]_{12} = .\dot{0}1\varphi 53\dot{7},$$
$$\tau(65) = (7 - 1)(\pi - 1) = 50,$$

where, as in all that follows, $\varphi = 10$, $\pi = 11$.

The number of places is 6, a divisor of $\tau(65)$.

$$\tfrac{1}{11} = [\tfrac{1}{13}]_8 - .\dot{0}56427213\dot{5}.$$

In this case the number of places $= \tau(13) = 12$.

8. Repetends in which the number of places in the period is equal to $\tau(N)$ are said to be perfect. This definition of a perfect repetend is equivalent to saying that $B$ is a primitive root of $N$, and viewed in this light the properties of primitive roots proved in Serret's *Cours d'Algèbre Supérieure*, Arts. 305, 317, 318, and 320 may be at once stated as properties of perfect repetends. They are embraced in the four theorems following: —

9. THEOREM V.—The fraction $N^{-1}$ cannot reduce to a perfect repetend unless $N = p^r$, $N = 2p^r$, or $N = 2^2$, where $p$ is an odd prime number.

10. THEOREM VI.—If $p$ is an odd prime number, and $p^{-r}$ reduce to a perfect repetend, $p^{-1}$ will also reduce to a perfect repetend.

11. THEOREM VII.—If $p^{-1}$, where $p$ is an odd prime number, reduce to a perfect repetend, $p^{-r}$ will or will not reduce to a perfect repetend, according as $B^{p-1} - 1$ is not, or is, divisible by $p^2$.

EXAMPLE: — $\frac{1}{7} = .\dot{1}4285\dot{7}$ is a perfect repetend and $10^{7-1} - 1$ is not divisible by $7^2 = 49$; therefore $7^{-n}$ will reduce to a perfect repetend. If $n = 2$, $7^{-2} = \frac{1}{49}$ $= .\dot{0}2040816326530612244897959183673469387755\dot{1}$, which is perfect, since $\tau(49)$ $= 7 \times 6 = 42$.

12. THEOREM VIII.—If $p^{-r}$, where $p$ is an odd prime number, reduce to a perfect repetend, so likewise does $\frac{1}{2}p^{-r}$.

13. If the base $B$ is an odd prime number we may determine *a priori* whether $N^{-1}$ reduces to a perfect repetend. By the preceding theorems it is only necessary to examine whether $p^{-1}$, where $p$ is an odd prime number, reduces to a perfect repetend. We will consider $p$ a prime number, and have

$$B^{(p-1)} - 1 \equiv 0, \quad \text{mod. } p;$$

$$\therefore \quad B^{\frac{1}{2}(p-1)} - 1 \equiv 0, \quad \text{mod. } p,$$

or

$$B^{\frac{1}{2}(p-1)} + 1 \equiv 0, \quad \text{mod. } p.$$

If

$$B^{\frac{1}{2}(p-1)} - 1 \equiv 0, \quad \text{mod. } p,$$

the repetend $p^{-1}$ is evidently not perfect.

14. It is usual to denote the positive or negative unit, which is the remainder of $B^{\frac{1}{2}(p-1)}$, mod. $p$, by the symbol $(B\,/\,p)$ which is known as "Legendre's symbol." We then have by the Law of Reciprocity

$$\left(\frac{B}{p}\right) = (-1)^{\frac{1}{2}(B-1)\frac{1}{2}(p-1)} \left(\frac{p}{B}\right).$$

And $(B/p) = +1$ is a sufficient condition that $p^{-1}$ should not reduce to a perfect repetend.

EXAMPLE: — Let $B = 7 = [10]_7$, $p = 29 = [41]_7$.

$$\left(\tfrac{10}{41}\right) = (-1)^{\frac{1}{2}(10-1)\ \frac{1}{2}(41-1)} \left(\tfrac{41}{10}\right) = \left(\tfrac{41}{10}\right).$$

It is evident, without expansion, that $41^{\frac{1}{2}(10-1)} \equiv 1$, mod. 10; therefore $\left(\tfrac{10}{41}\right) = +1$. Therefore the repetend is not perfect. By actual expansion $\frac{1}{41} = .\dot{0}14553\dot{6}$.

If $$B^{\frac{1}{2}(p-1)} + 1 \equiv 0, \quad \text{mod. } p,$$

the repetend will be perfect if $B$ does not satisfy a congruence,

$$B^m - 1 \equiv 0, \quad \text{mod. } p,$$

where $m$ is a divisor of $(p-1)$ and less than $\frac{1}{2}(p-1)$; in other words, if $p^{-1}$ does not reduce to a repetend of $m$ places.

EXAMPLE: — Let $B = 7 = [10]_7$, $p = 23 = [32]_7$.

$$\left(\tfrac{10}{32}\right) = (-1)^{\frac{1}{2}(10-1)\ \frac{1}{2}(32-1)} \left(\tfrac{32}{10}\right) = -\left(\tfrac{32}{10}\right).$$

But evidently $32^{\frac{1}{2}(10-1)} \equiv 1$, mod. 10,

$$\therefore \quad \left(\tfrac{10}{32}\right) = -1.$$

The only factor of $(32-1)$ less than $\frac{1}{2}(32-1)$ is 2; and since $B$ does not satisfy the congruence $B^2 - 1 \equiv 0$, mod. 32, the repetend $\frac{1}{32}$ is perfect. By expansion, $\frac{1}{32} = .\dot{0}20625113436460415532\dot{3}$.

15. The theory may be extended to the case of $B$, any odd number, by Jacobi's extension of the Law of Reciprocity. If $B = b_1 b_2 \ldots b_n$, where $b_1, b_2, \ldots b_n$ are the prime factors of $B$, Jacobi defines the symbol $(p/B)$ by the equation,

$$\left(\frac{p}{B}\right) = \left(\frac{p}{b_1}\right) \left(\frac{p}{b_2}\right) \cdots \left(\frac{p}{b_n}\right).$$

The following equation may then be established: —

$$\left(\frac{B}{p}\right) = (-1)^{\frac{1}{2}(B-1)\ \frac{1}{2}(p-1)} \left(\frac{p}{B}\right).$$

The conditions for determining the character of $p^{-1}$ are the same as those already given.

EXAMPLE: — Let $b_1 = 3$, $b_2 = 5$, $B = 15 = [10]_{15}$, $p = 7 = [7]_{15}$.

$$\left(\tfrac{10}{7}\right) = (-1)^{\frac{1}{2}(10-1)\ \frac{1}{2}(7-1)} \left(\tfrac{7}{10}\right) = -\left(\tfrac{7}{3}\right)\left(\tfrac{7}{5}\right).$$

But $\left(\tfrac{7}{3}\right) = +1$, $\left(\tfrac{7}{5}\right) = -1$, and therefore $\left(\tfrac{10}{7}\right) = +1$. Therefore the repetend $\frac{1}{7}$ is not perfect. In fact $\frac{1}{7} = .\dot{2}$.

16. The following are all the perfect repetends, $B = 10$, whose denominators are less than 100:—

$\frac{1}{7} = .\dot{1}4285\dot{7}$,

$\frac{1}{17} = .\dot{0}58823529411764\dot{7}$,

$\frac{1}{19} = .\dot{0}5263157894736842\dot{1}$,

$\frac{1}{23} = .\dot{0}43478260869565217391\dot{3}$,

$\frac{1}{29} = .\dot{0}34482758620689655172413793\dot{1}$,

$\frac{1}{47} = .\dot{0}21276595744680851063829787234042553191489361\dot{7}$,

$\frac{1}{49} = .\dot{0}2040816326530612244897959183673469387755\dot{1}$,

$\frac{1}{59} = .\dot{0}16949152542372881355932203389830508474576271186440677966\dot{1}$,

$\frac{1}{61} = .\dot{0}1639344262295081967213114754098360655737704918032786885245\dot{9}$,

$\frac{1}{97} = .\dot{0}10309278350515463917525773195876288659793814432989690721649484$
$536082474226804123711340206185567\dot{7}$.

$\frac{1}{98}$ is rejected although of the form $\frac{1}{2}p^{-r}$ because 2 is a divisor of 10, and therefore $\frac{1}{98}$ is not a pure repetend.

17. The following are all the perfect repetends, $B = 12 = [10]_{12}$, whose denominators are less than $50 = [42]_{12}$:—

$\frac{1}{5} = [\frac{1}{5}]_{12} = .\dot{2}49\dot{7}$,

$\frac{1}{7} = [\frac{1}{7}]_{12} = .\dot{1}86\varphi 3\dot{5}$,

$\frac{1}{17} = [\frac{1}{15}]_{12} = .\dot{0}857921 4\pi 36429\varphi\dot{7}$,

$\frac{1}{25} = [\frac{1}{21}]_{12} = .\dot{0}5915343\varphi 0\pi 62\varphi 68781\dot{\pi}$,

$\frac{1}{31} = [\frac{1}{27}]_{12} = .\dot{0}478\varphi\varphi 0935981 66\pi 74311\pi 28623\varphi 5\dot{5}$,

$\frac{1}{41} = [\frac{1}{35}]_{12} = .\dot{0}36190\varphi 653277397\varphi 9\pi 4\pi 85\varphi 2\pi 25689448241 20\dot{7}$,

$\frac{1}{43} = [\frac{1}{37}]_{12} = .\dot{0}342295\varphi 3\varphi\varphi 730\varphi 068456\pi 87992618 1148\pi 1\pi 5376\dot{5}$,

$\frac{1}{49} = [\frac{1}{41}]_{12} = .\dot{0}2\pi 322547\varphi 05\varphi 644\varphi 9380\pi 90899674 1\pi 6157712 83\dot{\pi}$.

18. THEOREM IX.—A number expressed by as many digits $(B - 1)$ as there are places in the repetend $N^{-1}$ is divisible by $N$.

For by Theorem IV, $B^n - 1$ is divisible by $N$.

EXAMPLES:— 999999 is divisible by 63 and also by 77; $\pi\pi\pi\pi\pi\pi$ is divisible by $[53]_{12}$ and also by $[65]_{12}$; 7777777777 is divisible by $[13]_8$.

19. Two numbers $M, M'$ are said to belong to the same group, if $MB^m \equiv M'$,

mod. $N$. If the repetend $N^{-1}$ is perfect, all the totitives of $N$ belong to the same group; but if the repetend $N^{-1}$ reduce to a repetend of $n$ places, where $n$ is a divisor of $\tau(N)$, the totitives of $N$ divide themselves into $\tau(N)/n$ groups of $n$ numbers each. Evidently,

20. THEOREM X.—The same repetend, by starting at different places, will express the value of all fractions having the same denominator, if the numerators belong to the same group.

21. THEOREM XI.—If in the process of reducing to a repetend, we arrive at a remainder $N - M$, we have half the repetend, and the remaining part may be found by subtracting each figure of the first part from $(B - 1)$.

For if $\dfrac{MB^k}{N} = q + \dfrac{N - M}{N} = (q + 1) - \dfrac{M}{N}$,

$$\frac{MB^{2k}}{N} = B^k(q + 1) + \frac{MB^k}{N} = (B^k - 1)(q + 1) + \frac{M}{N}. \quad Q.E.D.$$

Repetends which enjoy the property described in the preceding theorem are called complementary.

22. THEOREM XII.—Every repetend whose period consists of $2k$ places is complementary if the denominator of its generating fraction is prime.

For by Theorem IV,

$$B^{2k} - 1 \equiv 0, \quad \text{mod. } N;$$

i.e.

$$(B^k - 1)(B^k + 1) \equiv 0, \quad \text{mod. } N;$$

$$\therefore \quad B^k + 1 \equiv 0, \quad \text{mod. } N.$$

For if

$$B^k - 1 \equiv 0, \quad \text{mod. } N,$$

the period of $N$ consists of $k$ places instead of $2k$.

COROLLARY. All perfect repetends are complementary.

23. THEOREM XIII.—If a fraction reduce to a complementary repetend of $4k$ places, its denominator is the sum of two squares.

For in the proof of Theorem XI,

$$\frac{MB^{4k}}{N} = (B^{2k} - 1)(q + 1) + \frac{M}{N};$$

$$\therefore \quad M(B^{4k} - 1) = (B^{2k} - 1)(q + 1)N,$$

$$\frac{M(B^{2k} + 1)}{N} = q + 1.$$

Therefore $N$ is the sum of two squares, since it is a divisor of the sum of two squares.